Laminar Boundary-Layer Flows of Newtonian Fluids with Non-Newtonian Fluid Injectants

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The governing differential equations for the steady-state, laminar flow of a Newtonian fluid over a flat plate with a non-Newtonian fluid injected into the boundary layer at the surface of the plate are formulated to be amenable to solution using similarity concepts. The interface between the non-Newtonian fluid and the Newtonian fluid is assumed to be a streamline, along which solutions for the outer region and those for the inner region are matched. The governing equations are solved in terms of the similarity variables by transforming the boundary value problem into an initial value problem. The power-law model was used to relate the deformation and rate of strain tensors for the non-Newtonian injectant, and solutions are presented for both pseudoplastic and dilatant fluids.

Nomenclature

L = reference length
x = axial coordinate
y = normal coordinate
U_∞ = reference velocity (freestream)
$u(x,y)$ = axial component of velocity
$v(x,y)$ = normal component of velocity
ρ = density
μ = viscosity
τ_{xy} = shear stress
δ = distance from $y = 0$ to interface
η_δ = η_1 evaluated at δ
η_∞ = η_2 evaluated at freestream edge of boundary layer
$\bar{\eta}_1$ = normalized value of η_1; η_1/η_δ
$\bar{\eta}_2$ = normalized value of η_2; η_2/η_∞
η_1 = independent similarity variable for inner region, Eq. (21)
η_2 = independent similarity variable for outer region, Eq. (22)

Introduction

THE problem of determining the properties of the viscous flowfield formed on a body submerged in a flowing non-Newtonian fluid has become of increasing importance in recent years because of its various practical applications. Among initial theoretical investigations are those of Schowalter[1] and Acrivos, Shah, and Petersen[2] whose results first appeared in 1960. In these, the Prandtl boundary-layer hypotheses[3] and the concept of similar solutions[4] were applied to the equations of motion for the external, laminar flow of a power-law, non-Newtonian fluid.

Since these studies, a number of classical hydrodynamic problems wherein the flowfield is a non-Newtonian fluid have been investigated.[5–7] Several papers have also been published on the question of similar solutions and the types of inviscid flowfields consistent with the basic concept of similarity. These include Kapur and Srivastava,[8] Lee and Ames,[9] Berkovskii,[10] and Hansen and Na.[11]

Although they provide a technological base from which an understanding of the basic phenomena of non-Newtonian boundary layers may be obtained, the previous investigations have limited application to the injection of a non-Newtonian fluid into the flowfield at the body surface. Recently Thompson and Snyder[12,13] and Kim and Eraslan[14] presented numerical results for boundary layers with surface injection on two-dimensional and axisymmetric bodies submerged in a non-Newtonian fluid. These investigations were motivated by the experimental result that appreciable reductions in drag can be achieved by injecting non-Newtonian fluids into the boundary layer on a submerged body.[15]

The present paper concerns injecting a non-Newtonian fluid into the boundary layer on a body submerged in a Newtonian fluid. The freestream and the outer region of the boundary layer are characterized by a Newtonian shear stress behavior, and the inner region of the boundary layer adjacent to the body surface by a non-Newtonian behavior. The boundary-layer equations are formulated to be amenable to similar solutions for the flow of a Newtonian fluid past a semi-infinite flat plate with a non-Newtonian fluid injected into the boundary layer at its surface. The fluids are incompressible and zero mass transfer across the interface between the inner and outer regions is assumed.

Mathematical Formulation

The Navier-Stokes equations for laminar flow of constant property fluids, without dissipation, past a flat plate can be simplified by using Prandtl's boundary-layer hypotheses.[3] The governing differential equations for the inner region are

Continuity

$$\partial u_1/\partial x + \partial v_1/\partial y = 0 \quad (1)$$

Momentum

$$\rho_1 u_1 \partial u_1/\partial x + \rho_1 v_1 \partial u_1/\partial y = (\partial/\partial y)(\tau_{xy}) \quad (2)$$

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Subscript 1 denotes inner region variables and the 2 denotes outer region. The shear stress for a power-law, non-Newtonian fluid has the form,[16]

$$\tau_{xy} = K(\partial u_1/\partial y)^N \quad (3)$$

where K and N are empirical constants.

The governing equations for the outer region are

Continuity

$$\partial u_2/\partial x + \partial v_2/\partial y = 0 \quad (4)$$

Momentum

$$\rho_2 u_2 \partial u_2/\partial x + \rho_2 v_2 \partial u_2/\partial y = \mu \partial^2 u_2/\partial y^2 \quad (5)$$

The usual conditions of zero tangential velocity and specified normal velocity at the wall, together with the tangential velocity equal to the freestream velocity at the outer edge of the boundary layer, are used to obtain wall and freestream boundary conditions for the viscous boundary layer. Expressed mathematically,

$$u_1(x,0) = 0,\ v_1(x,0) = v_s(x) \qquad \lim_{y\to\infty} u_2(x,y) = U_\infty \quad (6)$$

The mathematical statement is completed by specifying appropriate matching conditions at the inner-outer region interface. Sparrow et al.[17] have considered the effect of injecting air into water and shown that within the framework of the boundary-layer assumptions the interface matching conditions are obtained by considering 1) the conservation of mass, 2) the tangential velocity as being continuous, and 3) the tangential shear as being continuous.

The interfacial velocity vector and the unit normal vector are given by

$$\mathbf{V} = u\mathbf{i} + v\mathbf{j}$$

and

$$\mathbf{n} = (dx\mathbf{j} - dy\mathbf{i})/(dx^2 + dy^2)^{1/2} \quad (7)$$

Thus, the mass transfer across the interface becomes

$$\rho \mathbf{V}\cdot\mathbf{n} = \rho(vdx - udy)/(dx^2 + dy^2)^{1/2} \quad (8)$$

The remaining analysis is restricted to cases where the curvature of the interface is small enough that $(dy/dx)^2 \ll 1$.

The first interface matching condition, obtained from mass conservation, is

$$\rho_1[v_1(x,y) - u_1(x,y)dy/dx]_{y=\delta} = \rho_2[v_2(x,y) - u_2(x,y)dy/dx]_{y=\delta} \quad (9)$$

where δ denotes distance from the wall to the inner-outer region interface.

The matching conditions for the tangential velocity and shear are formally expressed as

$$u_1(x,y)|_{y=\delta} = u_2(x,y)|_{y=\delta} \quad (10)$$

and

$$K(\partial u_1/\partial y)^N|_{y=\delta} = \mu(\partial u_2/\partial y)|_{y=\delta} \quad (11)$$

Equations (1–11) provide a complete mathematical description, within the framework of boundary-layer theory, of the physical phenomena throughout the boundary layer.

Introducing nondimensional velocity and space variables,

$$\bar{u}(\bar{x},\bar{y}) = u(x,y)/U_\infty,\ \bar{v}(\bar{x},\bar{y}) = [v(x,y)/U_\infty](Re_{L_1})^{1/(1+N)} \qquad \bar{x} = x/L,\ \bar{y} = (y/L)(Re_{L_1})^{1/(1+N)} \quad (12)$$

where $Re_{L_1} = \rho_1 U_\infty^{(2-N)}L^N/K$ is the generalized Reynolds number,[18] into Eqs. (1) and (2) yields their nondimensional form.

The well-known nondimensional form of Eqs. (4) and (5) are obtained using Eq. (12) with $N = 1$, $K = \mu$, and all barred quantities formally replaced with tilded quantities.

Thus, the mathematical system, in terms of nondimensional quantities, can be stated as

The Inner Region

$$\partial\bar{u}/\partial\bar{x} + \partial\bar{v}/\partial\bar{y} = 0 \quad (13)$$

$$\bar{u}\partial\bar{u}/\partial\bar{x} + \bar{v}\partial\bar{u}/\partial\bar{y} = (\partial/\partial\bar{y})(\partial\bar{u}/\partial\bar{y})^N \quad (14)$$

The Outer Region

$$\partial\tilde{u}/\partial\tilde{x} + \partial\tilde{v}/\partial\tilde{y} = 0 \quad (15)$$

$$\tilde{u}\partial\tilde{u}/\partial\tilde{x} + \tilde{v}\partial\tilde{u}/\partial\tilde{y} = \partial^2\tilde{u}/\partial\tilde{y}^2 \quad (16)$$

with boundary conditions,

$$\bar{u}(\bar{x},0) = 0,\ \bar{v}(\bar{x},0) = [v_s(x)/U_\infty](Re_{L_1})^{1/(1+N)} \qquad \lim_{\tilde{y}\to\infty} \tilde{u}(\tilde{x},\tilde{y}) = 1 \quad (17)$$

and interfacial matching conditions,

$$\frac{\rho_1}{(Re_{L_1})^{1/(1+N)}}\left[\bar{v}(\bar{x},\bar{y}) - \bar{u}(\bar{x},\bar{y})\frac{d\bar{y}}{d\bar{x}}\right] = \frac{\rho_2}{(Re_{L_2})^{1/2}}\left[\tilde{v}(\tilde{x},\tilde{y}) - \tilde{u}(\tilde{x},\tilde{y})\frac{d\tilde{y}}{d\tilde{x}}\right] \quad (18)$$

$$\bar{u}(\bar{x},\bar{y}) = \tilde{u}(\tilde{x},\tilde{y}) \quad (19)$$

$$(KU_\infty{}^N/L^N)(Re_{L_1})^{N/(1+N)}(\partial\bar{u}/\partial\bar{y})^N = (\mu U_\infty/L)(Re_{L_2})^{1/2}\,\partial\tilde{u}/\partial\tilde{y} \quad (20)$$

where all variables and derivatives in the matching conditions are evaluated at the inner-outer region interface.

The next step is to apply similarity analyses[4] to reduce the number of independent variables and obtain ordinary (rather than partial) differential equations. The similarity variables for the inner region are defined as

$$f'(\eta_1) = u(x,y)/U_\infty;\quad \eta_1 = y/L(L/2x)^{1/(1+N)}(Re_{L_1})^{1/(1+N)} \quad (21)$$

and those for the outer region as

$$f'(\eta_2) = u(x,y)/U_\infty;\quad \eta_2 = [(y-\delta)/L]\times(L/2x)^{1/2}(Re_{L_2})^{1/2} \quad (22)$$

where primes denote differentiation with respect to η.

Applying Eqs. (21) and (22) to Eqs. (13-16) yields the following transformed governing differential equations

The Inner Region

$$f'''(\eta_1) + [2/N(1+N)]f(\eta_1)[f''(\eta_1)]^{2-N} = 0 \quad (23)$$

The Outer Region

$$f'''(\eta_2) + f(\eta_2)f''(\eta_2) = 0 \quad (24)$$

The boundary conditions, Eq. (17), written in terms of the similarity variables, become

$$f'(\eta_1) = 0,\ f(\eta_1) = f(0),\ @\ \eta_1 = 0 \qquad \lim_{\eta_2\to\infty} f'(\eta_2) = 1 \quad (25)$$

where

$$f(0) = -(1+N)(Re_{L_1}/2)^{1/(1+N)}(x/L)^{N/(1+N)}v_s(x)/U_\infty$$

is the surface mass transfer parameter. Mass injection corresponds to $f(0) < 0$ and is the only meaningful situation for the postulated flow configuration.

The transformed matching conditions at the interface are

$$(\rho_1/1+N)(2/Re_{L_1})^{1/(1+N)}(L/x)^{N/(1+N)}f(\eta_1) = (\rho_2/2)(2/Re_{L_2})^{1/2}(L/x)^{1/2}f(\eta_2) \quad (26)$$

$$f'(\eta_1) = f'(\eta_2) \quad (27)$$

$$(KU_\infty{}^N/L^N)(Re_{L_1})^{N/(1+N)}(L/2x)^{N/(1+N)}[f''(\eta_1)]^N = \mu U_\infty/L(Re_{L_2})^{1/2}(L/2x)^{1/2}f''(\eta_2) \quad (28)$$

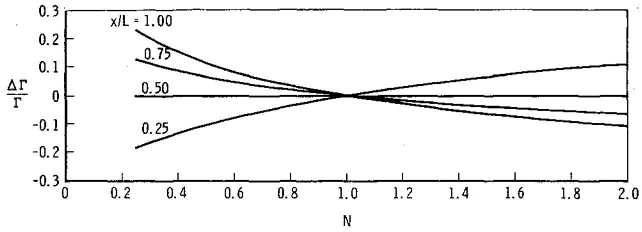

Fig. 1 Variation of Γ with x/L.

Equation (26) accounts for mass transfer between the inner and outer regions, as might occur due to mixing or evaporation at the interface. If the postulate is made, as has been done here, that because of the laminar nature of the flow the interface is a streamline, then Eq. (26) accounts for all interfacial mass transfer and can be written in the simple, convenient form,

$$f(\eta_1) = f(\eta_2) = 0, @\ y = \delta \tag{29}$$

Eq. (29) can be shown to be the correct matching condition by considering a mass balance for the conditions indicated earlier. The integral form of this mass balance is

$$\int_0^x \rho_1 v_s(x)dx = \int_0^\delta \rho_1 u(x,y)dy$$

Transforming to the η_1 similarity plane and performing the integration yields $f(\eta_1) = 0$ at $y = \delta$. Eq. (29) is then obtained using this result in Eq. (26).

Examination of Eqs. (23) and (24), keeping in mind the definitions of the independent similarity variables and the interface matching conditions, reveals that the momentum equation for the inner region, Eq. (23), is completely similar whereas the necessity for specifying x/L in the interface matching conditions leads to a condition of local similarity for the momentum equation in the outer region. Although of less convenience than where self-similarity prevails throughout, the computation time required is small, and several values of x/L can be investigated without excessive computation. For the special case $N = 1$, $\rho_1 \neq \rho_2$ and μ_1 μ_2 (i.e., $K = \mu$, for $N = 1$) Eqs. (23) and (24), with associated boundary conditions and interface matching conditions, reduce to those given in Ref. 17.

Numerical Solution

The technique of transforming a two point boundary-value problem into an equivalent initial value problem,

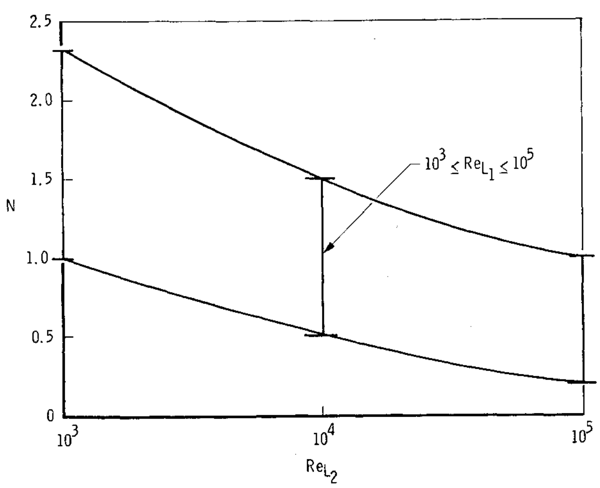

Fig. 2 Range of N covered by numerical calculations.

Table 1 Comparison of present results for air injected into water with results of Sparrow et al.

Γ	$f(0)$	$f''(0)^a$	$f'(\delta)^a$	δ^a	$f''(0)^b$	$f'(\delta)^b$	δ^b
0.005	−0.30	1.476	0.989	0.619	1.585	0.983	0.620
0.008	−0.05	8.316	0.913	0.109	7.820	0.884	0.113
0.0133	−0.10	4.241	0.926	0.216	4.020	0.900	0.222
0.040	−0.20	1.906	0.895	0.461	1.805	0.862	0.467

[a] Present results.
[b] Sparrow et al., Ref. 17.

first given by Töpfer[19] and more recently reconsidered by Klamkin,[20] can be applied to obtain solutions for Eqs. (23) and (24). The basic motivation in attempts to obtain an equivalent initial value problem is the elimination of the requirement for some type of convergence scheme to insure satisfaction of the boundary conditions at $\eta_2 \rightarrow \infty$ when the problem is solved as a two-point boundary-value problem.

Transformation variables F and G are defined by

$$f(\eta_1) = \alpha^a F(\alpha^b \eta_1) \tag{30}$$

and

$$f(\eta_2) = \beta^{1/3} G(\beta^{1/3}\eta_2) \tag{31}$$

The constants a and b, determined by substituting Eq. (30) into Eq. (23) and requiring the value of the resulting exponent of α be zero, are

$$a = (2N - 1)/3(2 - N),\ N \neq 2 \qquad b = \tfrac{1}{3} \tag{32}$$

Constants α and β are determined in the course of solving the transformed equations,

$$F'''(z_1) + [2/N(1 + N)]F(z_1)[F''(z_1)]^{2-N} = 0 \tag{33}$$

and

$$G'''(z_2) + G(z_2)G''(z_2) = 0 \tag{34}$$

where

$$F(z_1) = F(\alpha^{1/3}\eta_1)$$

and

$$G(z_2) = G(\beta^{1/3}\eta_2)$$

The boundary conditions for Eqs. (33) and (34) can be written as

$$F(0) = \frac{f(0)}{(\alpha)^{(2N-1)/[3(2-N)]}}, \quad G(0) = 0$$

$$F'(0) = 0, \qquad G'(0) = \left[\frac{(\alpha)^{(1+N)/[3(2-N)]}}{\beta^{2/3}}\right] F'(\delta)$$

$$F''(0) = 1, \qquad G''(0) = \Gamma\left[\frac{(\alpha)^{N/(2-N)}}{\beta}\right][F''(\delta)]^N \tag{35}$$

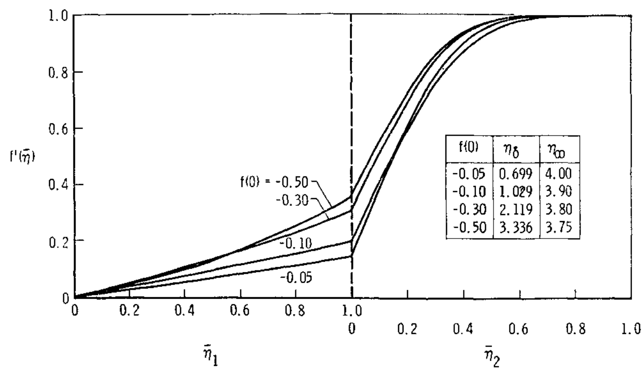

Fig. 3 Velocity profiles for $N = 0.50$.

Table 2 Variations of skin-friction coefficient with Γ

N	x/L	$\Delta\Gamma/\Gamma$	$\Delta C_f/C_f$
0.25	1.00	0.23	−0.0352
0.50	0.25	−0.11	0.0212
1.50	0.25	0.07	−0.0081
2.00	1.00	−0.11	0.0055

Γ is determined from Eq. (28) and the definitions of $F(z_1)$ and $G(z_2)$, as

$$\Gamma = (\rho_1/\rho_2)(Re_{L_2})^{1/2}/(Re_{L_1})^{1/(1+N)}\ (2x/L)^{(1-N)/2(1+N)} \quad (36)$$

Specified values for $F(0)$, $G'(0)$ and $G''(0)$ were used to obtain a solution for Eqs. (33) and (34) and then used in an iteration to obtain predetermined values of $f(0)$ and Γ.

Equation (33) was integrated from $\eta_1 = 0$ to the value of η_1 at which $f(\eta_1) = 0$. Equation (34) was then integrated from $\eta_2 = 0$; i.e., the interface, to the value of η_2 at which the variation in $G'(z_2)$ was less than a certain specified value. Since

$$f'(\eta_2) = \beta^{2/3}G'(z_2) \text{ and } \lim_{\eta\to\infty} f'(\eta_2) = 1$$

the required value of β was obtained from the limiting value of $G'(z_2)$,

$$\beta = [1/\lim_{\eta_2\to\infty} G'(z_2)]^{3/2} \quad (37)$$

Thus, the equations for $F(0)$, $G'(0)$, $G''(0)$ and Eq. (37) provide four equations for the four unknowns $f(0)$, α, Γ, and β.

Having obtained the solutions of Eqs. (33) and (34) consistent with the desired values of $f(0)$ and Γ, Eqs. (23) and (24) can be written as,

$$f'''(\eta_1) + [2/N(1+N)]f(\eta_1)[f''(\eta_1)]^{(2-N)} = 0 \quad (38)$$

with boundary conditions

$$f(0) = f(0), f'(0) = 0, f''(0) = (\alpha)^{1/2-N}$$

and

$$f'''(\eta_2) + f(\eta_2)f''(\eta_2) = 0 \quad (39)$$

with boundary conditions

$$f(0) = 0, f'(0) = f'(\eta_1)|_{\eta_1=\eta_\delta}$$
$$f''(0) = \Gamma|[f''(\eta_1)|_{\eta_1=\eta_\delta}]^N$$

The solution of Eq. (38) is terminated at $f(\eta_1) = 0$ and that of Eq. (39) when the condition

$$\lim_{\eta_2\to\infty} f'(\eta_2) = 1$$

is sufficiently approximated. Thus, the task of solving two two-point boundary-value problems, Eqs. (23) and (24), has been reduced to the less difficult task of solving four initial-value problems, Eqs. (33, 34, 38, and 39).

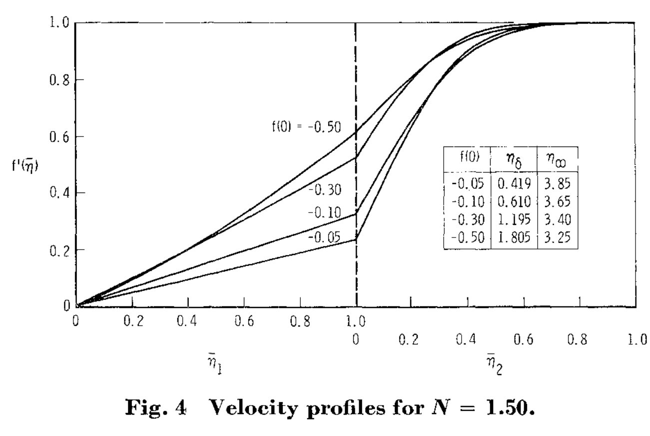

Fig. 4 Velocity profiles for N = 1.50.

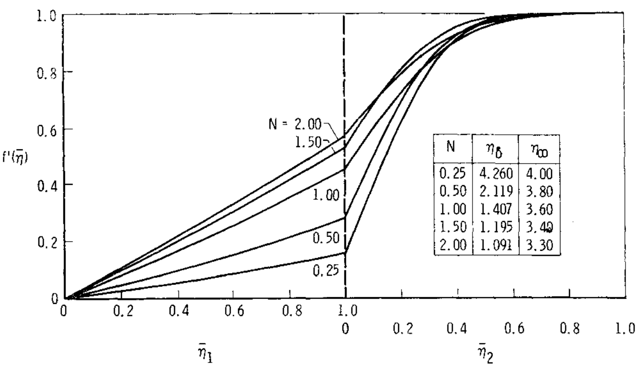

Fig. 5 Velocity profiles for 0.25 ≤ N ≤ 2.0 and $f(0)$ = −0.30.

In addition to the velocity profiles, the more useful boundary-layer quantities such as shear and displacement thickness coefficients were obtained from the following equations:

Displacement thickness

$$\frac{\delta^*}{x}\left(\frac{Re_{x_2}}{2}\right)^{1/2} = \Gamma\left(\frac{\rho_2}{\rho_1}\right)\int_0^{\eta_\delta}\left[1 - \left(\frac{\rho_1}{\rho_2}\right)f'(\eta_1)\right]d\eta_1 + \int_0^\infty [1 - f'(\eta_2)]d\eta_2 \quad (40)$$

Momentum thickness

$$\frac{\theta}{x}\left(\frac{Re_{x_2}}{2}\right)^{1/2} = \Gamma\left(\frac{\rho_2}{\rho_1}\right)\int_0^{\eta_\delta}\left[1 - \left(\frac{\rho_1}{\rho_2}\right)f'(\eta_1)\right]f'(\eta_1)d\eta_1 + \int_0^\infty [1 - f'(\eta_2)]f'(\eta_2)d\eta_2 \quad (41)$$

Skin-friction coefficient

$$C_{f_x}(Re_{x_2}/2)^{1/2} = \Gamma[f''(\eta_1)|_{\eta_1=0}]^N \quad (42)$$

Discussion of Numerical Results

Table 1 compares results obtained by Sparrow et al.[17] using an approximate technique to solve the governing equations for air injected into water, and the results for the same problem obtained as a special case of the present formulation; i.e., $N = 1$, $K = \mu_1$. Although some discrepancies exist, the over-all agreement is seen to be rather good. A literature survey failed to reveal comparable data with which the present results for the non-Newtonian injectant could be compared. However, the comparison in Table 1, being a specialized case of the more general problem, seems to indicate the validity of the present method.

Numerical calculations were made for ranges of the shear stress exponent, N, 0.25 ≤ N ≤ 2.0 and the mass transfer

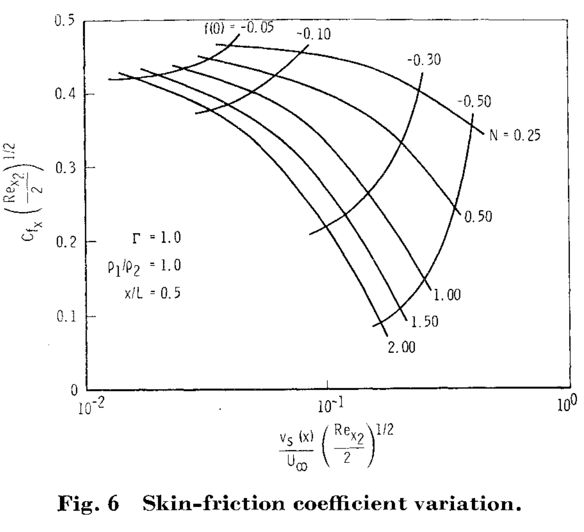

Fig. 6 Skin-friction coefficient variation.

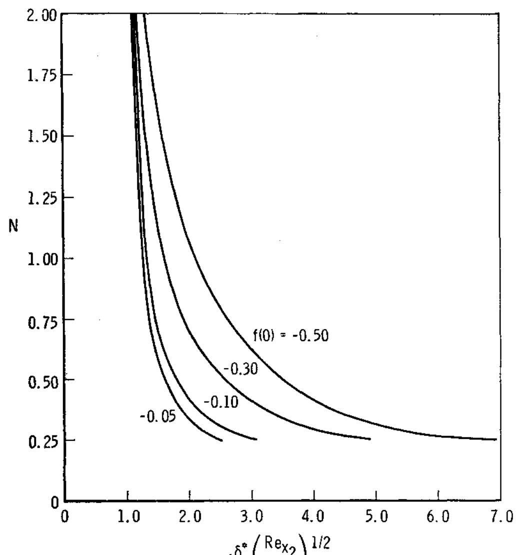

Fig. 7 Displacement thickness variation.

parameter, $f(0)$, $-0.05 \leq f(0) \leq -0.50$. The parameter Γ was taken as unity. Since the density ratio, ρ_1/ρ_2, can be considered near unity when the freestream is water[18] and the Reynolds numbers are constant for a given situation, the variation in Γ with x/L can be obtained from Eq. (36). As shown in Fig. 1 the maximum variation occurs at $x/L = 1.0$ and $N = 0.25$ and amounts to a 23% change in Γ. The effect of these variations in Γ on the numerical results is shown in Table 2 in terms of the variation of the skin friction coefficient, and is small for all cases. Assuming $\Gamma = 1$, a fixed value of x/L, say 0.5, and a range of freestream Reynolds number, $10^3 \leq Re_{L_2} \leq 10^5$, the applicability of the present results can be estimated using the criteria that the generalized Reynolds number, Re_{L1}, also be in the range $10^3 \leq Re_{L1} \leq 10^5$ (see Refs. 13 and 20). The range of N as a function of Re_{L_2} for which the present results are applicable is shown in Fig. 2.

Velocity profiles for $N = 0.5$ and 1.50 are shown in Figs. 3 and 4 as a function of the normalized distances $\bar{\eta}_1$ and $\bar{\eta}_2$ and the mass transfer parameter $f(0)$. The velocity at the interface is greater for $N = 1.5$ than for $N = 0.5$. As expected, increased mass transfer increases the thickness of the inner region.

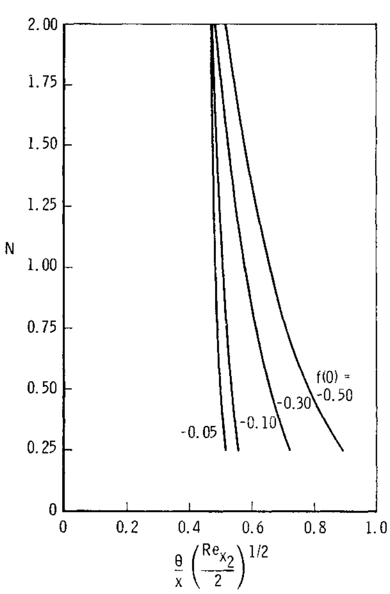

Fig. 8 Momentum thickness variation.

Velocity profiles for $0.25 \leq N \leq 2.0$ are shown in Fig. 5 for a constant value of the mass transfer parameter, $f(0) = -0.30$.

The skin-friction coefficient is shown in Fig. 6. In general its change is greater for a given change in mass injection than for a corresponding change in the shear exponent. The displacement and momentum thickness parameters are shown in Figs. 7 and 8, respectively. The variation in these parameters with mass injection is greatest at $N = 0.25$ and decreases significantly at $N = 2.0$, apparently due to the decreased influence of the first terms on the right side of Eqs. (40) and (41) as N increases.

Summary

The governing set of equations for the boundary-layer flow over a body can be formulated using similarity concepts for the case of a semi-infinite flat plate submerged in a Newtonian fluid with a power-law, non-Newtonian fluid injected into the boundary layer. Although the transformed equations are not similar in the strictest sense, the condition which prevents complete similarity; i.e., x/L in the parameter Γ, is a relatively weak one permitting the use of the concept of local similarity.

In general, the numerical results indicate that the inner region of the boundary layer becomes thicker with increased mass injection at constant N and thinner with increased N at constant mass injection. The outer region becomes thinner for both cases. The behavior of the displacement thickness as N is varied indicates that the inner region thickening and the outer region thinning nearly cancel for dilatant injectants and the inner region thickening dominates for pseudoplastic injectants.

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